SYLVAIN KAHANE

On the complexity of sums of Dirichlet measures


<http://www.numdam.org/item?id=AIF_1993__43_1_111_0>
ON THE COMPLEXITY OF SUMS OF DIRICHLET MEASURES

by Sylvain KAHANE

1. Introduction.

Let $E$ be a metrizable compact space. We denote by $\mathcal{M}(E)$ (resp. $\mathcal{M}_1(E)$) the set of all non-negative (resp. probability) Borel measures on $E$. Recall that $\mathcal{M}_1(E)$ is a metrizable compact space for the weak* topology, which is the topology of the duality with the set $\mathcal{C}(E)$ of all continuous functions on $E$; in the following, the topological complexity of a subset $M$ of $\mathcal{M}(E)$ actually means the topological complexity of $M \cap \mathcal{M}_1(E)$. $\mathcal{P}(E)$ (resp. $\mathcal{K}(E)$) denotes the set of all subsets (resp. compact subsets) of $E$. Let $C$ be a closed under countable intersections subset of $\mathcal{P}(E)$. We denote by $\mathcal{M}(C)$ the set of all non-negative Borel measures concentrated on an element of $C : \mathcal{M}(C) = \bigcup_{X \in C} \mathcal{M}(X)$. Let $C^\sigma$ (resp. $C^\uparrow$) denote the set of all unions of sequences (resp. increasing sequences) of elements of $C$. Note that $\mathcal{M}(C^\uparrow)$ is equal to the norm-closure $\mathcal{M}(C)$ of $\mathcal{M}(C)$ and that $\mathcal{M}(C^\sigma)$ is the convex norm-closure of $\mathcal{M}(C)$. We denote $C^\perp$ the set of all measures which annihilate all elements of $C$. We have the following algebraic decomposition : $\mathcal{M}(E) = \mathcal{M}(C^\sigma) \oplus C^\perp$. Recall that $\mathcal{K}(E)$ is a metrizable compact space in the Hausdorff topology. If $C$ is a Borel subset of $\mathcal{K}(E)$, then $\mathcal{M}(C)$, $\mathcal{M}(C^\uparrow)$ and $\mathcal{M}(C^\sigma)$ are analytic subsets of $\mathcal{M}_1(E)$ and $C^\perp$ is a coanalytic subset of $\mathcal{M}_1(E)$.

Let $T$ be the unit circle $\mathbb{R}/\mathbb{Z}$. We are interested in the four following subsets of $\mathcal{K}(T)$.

Key words : Analytic sets — Dirichlet measures — Singular measures — Sums of measures. A.M.S. Classification : 28A33 — 04A15.
A compact subset $K$ of $\mathbf{T}$ is a set of type $D$ or a Dirichlet set if for all $\varepsilon > 0$ and $N \in \mathbf{N}$ there exists $n \geq N$ such that $|\sin 2\pi nx| < \varepsilon$ for all $x \in K$.

A compact subset $K$ of $\mathbf{T}$ is a set of type $H$ if there exist a non empty interval $I$ of $\mathbf{T}$ and a strictly increasing sequence $(n_k)_{k \in \mathbf{N}}$ of integers such that $n_k K \cap I = \emptyset$ for each integer $k$.

A compact subset $K$ of $\mathbf{T}$ is a set of type $L$ or a lacunary set if there exist a sequence $\varepsilon_n \to 0^+$, a sequence $\alpha_n \to +\infty$ and for each integer $n$ a finite sequence $(I_k)$ of intervals such that $|I_k| \leq \varepsilon_n$ for each $k$, $d(I_k, I_{k'}) \geq \alpha_n \varepsilon_n$ for each $k \neq k'$ and $K \subseteq \bigcup I_k$.

A compact subset $K$ of $\mathbf{T}$ is a set of type $L_0$ if there exist a sequence $\varepsilon_n \to 0^+$, $\alpha > 0$ and for each integer $n$ a finite sequence $(I_k)$ of intervals such that $|I_k| \leq \varepsilon_n$ for each $k$, $d(I_k, I_{k'}) \geq \alpha \varepsilon_n$ for each $k \neq k'$ and $K \subseteq \bigcup I_k$.

Note that both $H$ and $L$ are supersets of $D$ and subsets of $L_0$. The classes $D$ and $L$ are $\mathcal{G}_\delta$ subsets of $\mathcal{K}(\mathbf{T})$ and $H$ and $L_0$ are $\mathcal{K}_{\sigma\delta}$ subsets [1].

A measure concentrated on a $D^\dagger$-set is called a Dirichlet measure. For every $\mu \in \mathcal{M}(\mathbf{T})$ and $n \in \mathbf{N}$, we denote $\hat{\mu}(n) = \int e^{2\pi ix} \, d\mu(x)$ and $\tilde{\mu}(n) = \int |\sin 2\pi nx| \, d\mu(x)$. For every $\mu \in \mathcal{M}(\mathbf{T})$, the following conditions are equivalent:

\[
\begin{cases}
(1) \mu \in \mathcal{M}(D^\dagger) \\
(2) \limsup_{n \to \infty} |\hat{\mu}(n)| = \int d\mu \\
(3) \liminf_{n \to \infty} \tilde{\mu}(n) = 0.
\end{cases}
\]

Note that $\mathcal{M}(D^\dagger)$ is a norm-closed $\mathcal{G}_\delta$ subset of $\mathcal{M}_1(\mathbf{T})$.

**Theorem 1.1.** — There does not exist a Borel subset $B$ of $\mathcal{M}_1(\mathbf{T})$ such that $B \cap L_0^\perp = \emptyset$ and $\mathcal{M}(D^\dagger) + \mathcal{M}(D^\dagger) \subset B$.

For all $M \subseteq \mathcal{M}(\mathbf{T})$ and $n \in \mathbf{N}$, we denote $M^{(n)}$ the set of all sums of $n$ elements of $M$.

**Corollary 1.2.** — The sets $\mathcal{M}(C^\dagger)^{(n)}$, $\mathcal{M}(C^\dagger)^{(n)}$ and $\mathcal{M}(C^\sigma)$ are analytic non Borel for all $n \geq 2$ and $C = D$, $H$, $L$ or $L_0$.

We obtain also the following property which has been studied successively by Host, Louveau and Parreau [3], Kechris and Lyons [3] and Kaufman [2].
Corollary 1.3. — $C^\perp$ is a coanalytic non Borel set for $C = D, H, L$ or $L_0$.

Corollary 1.4. — None of the sets in the two previous corollaries can be pairwise separated by a Borel set.

We prove also that the sets $\mathcal{M}(C^\perp)^{(n)}$, for $n \geq 2$ and $C = D, H, L$ or $L_0$, are not norm-closed.

Theorem 1.5. — There exists a measure in $\mathcal{M}(D^\perp) + \mathcal{M}(D^\perp)$ which is not a finite sum of measures in $\mathcal{M}(L_0^\perp)$.

Theorem 1.6. — For every $n \geq 3$, there exists a measure in $\mathcal{M}(D^\perp) + \mathcal{M}(D^\perp)$ which is the sum of $n$ measures in $\mathcal{M}(D^\perp)$ and is not the sum of $n-1$ measures in $\mathcal{M}(L_0^\perp)$.

2. Kaufman’s reduction.

We follow Kaufman’s construction used to prove that $H^\perp$ is not a Borel set [2]. Let $\mathbb{N}$ be the set of positive integers, $[\mathbb{N}]$ be the set of all infinite subsets of $\mathbb{N}$, $\mathbb{N}^{<\mathbb{N}}$ be the set of all finite sequences of positive integers and $T$ be the set of trees on $\mathbb{N}$, i.e., $T \subseteq \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ and $T \in T$ if and only if all initial segments of $s \in T$ are also in $T$. We say that $T \in T$ is a well founded tree if $T$ has no infinite branch, i.e., there does not exist $\sigma \in \mathbb{N}^{\mathbb{N}}$ all whose initial segments belong to $T$. The set of all well founded trees is denoted by $WF$. Recall that $T$ is a Polish space in the product topology on $\mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ and $WF$ is the classical example of a coanalytic non Borel set.

We denote $2^{\mathbb{N}}$ the compact, metrizable space $\{0, 1\}^{\mathbb{N}}$. If $x \in 2^{\mathbb{N}}$, $x = (x(n))_{n \in \mathbb{N}}$ with $x(n) = 0$ or $1$. Let $\lambda$ be the Lebesgue measure on $2^{\mathbb{N}}$. Let $\Sigma$ be the Polish space of all Borel sets on $2^{\mathbb{N}}$ with metric $d(A, B) = \lambda(A \triangle B)$, quotiented by the relation $d(A, B) = 0$; $\Sigma$ can be viewed as a closed subspace of $L^1(2^{\mathbb{N}})$. Consider the sets

$\mathcal{X} = \{ (A_n)_{n \in \mathbb{N}} \in \Sigma^{\mathbb{N}}; \lambda(\bigcap_{R} A_R) = 0 \text{ for all } R \in [\mathbb{N}] \}$

and

$\mathcal{Y} = \{ (A_n)_{n \in \mathbb{N}} \in \Sigma^{\mathbb{N}}; \lambda(\liminf_{R} A_n \cup \liminf_{S} A_n) = 1 \text{ for some } (R, S) \in [\mathbb{N}]^2 \}$,
where $\liminf_R A_n = \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m, n \in R} A_n$. Note that $\mathcal{X}$ is a coanalytic subset of $\Sigma^N$ [2] and that $\mathcal{Y}$ is an analytic subset of $\Sigma^N$.

**Lemma 2.1.** — There is a continuous mapping $\Phi$ from $T$ to $\Sigma^N$ such that $\Phi(WF) \subset \mathcal{X}$ and $\Phi(WF^c) \subset \mathcal{Y}$. Therefore, there is no Borel subset $B$ of $\Sigma^N$ such that $\mathcal{Y} \subset B$ and $\mathcal{X} \cap B = \emptyset$.

**Proof.** — **Construction of $\Phi$.** To each $s \in \mathbb{N}^N$, we attach subsets $E(s)$ and $F(s)$. Let $<, >$ be a one-to-one mapping from $\mathbb{N}^2$ to $\mathbb{N}$. We define $E(s)$ and $F(s)$ by induction on the length $|s|$ of $s$. Let $E(\emptyset) = 2^N$ and $F(\emptyset) = \emptyset$. If $s \in \mathbb{N}^N$ has length $|s| = k - 1$ and $n_k \in \mathbb{N}$, put

$$E(s^{\sim}n_k) = \{ x \in 2^N; (x \in E(s) \text{ and } \exists i \in [kn_k, kn_k + 1), x(<k, i>) = 0) \}
\quad \text{or} \quad (x \in F(s) \text{ and } \forall i \in [kn_k, kn_k + 1), x(<k, i>) = 1) \}$$

and

$$F(s^{\sim}n_k) = \{ x \in 2^N; (x \in F(s) \text{ and } \exists i \in [kn_k, kn_k + 1), x(<k, i>) = 0) \}
\quad \text{or} \quad (x \in E(s) \text{ and } \forall i \in [kn_k, kn_k + 1), x(<k, i>) = 1) \}.$$

We have $E((n_1)) = \{ x \in 2^N; x(<1, n_1>) = 0 \}$ and $F((n_1)) = \{ x \in 2^N; x(<1, n_1>) = 1 \}$ if $n_1 \in \mathbb{N}$. Note that $E(s) = F(s)^c$ and $\lambda(E(s)) = \lambda(F(s)) = \frac{1}{2}$ for all $s \in \mathbb{N}^N \setminus \{\emptyset\}$. Let $\sigma \in \mathbb{N}^N$. The length $k$ initial segment of $\sigma$ is denoted by $\sigma_{\downarrow k}$. We have

$$\lambda\left( \bigcap_{k \geq n} E(\sigma_{\downarrow k}) \right) \geq \lambda(E(\sigma_{\downarrow n})) \times \prod_{k > n} (1 - 2^{-k})$$

for each $n \in \mathbb{N}$. But $\lim_{n \to +\infty} \prod_{k > n} (1 - 2^{-k}) = 1$, whence

$$\lambda\left( \liminf E(\sigma_{\downarrow k}) \cup \liminf F(\sigma_{\downarrow k}) \right) = 1.$$

Let us enumerate $\mathbb{N}^N = \{ s_n; n \in \mathbb{N} \}$ and consider the mapping $\Phi : T \to \Sigma^N, T \mapsto (\Phi_n(T))_{n \in \mathbb{N}}$ defined by

$$\Phi_n(T) = \begin{cases} E(s_p) & \text{if } n = 2p \text{ and } s_p \in T, \\ F(s_p) & \text{if } n = 2p + 1 \text{ and } s_p \in T, \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly, $\Phi$ is continuous and $\Phi(WF^c) \subset \mathcal{Y}$.

To complete the proof of Lemma 2.1, it remains only to show that $\Phi(WF) \subset \mathcal{X}$. Let $T \in T$ such that there exists $R \in [\mathbb{N}]$ with $\lambda\left( \bigcap_{R} \Phi_n(T) \right) > 0$. Let us suppose that $R \cap 2\mathbb{N}$ is infinite (the case
\( R \cap (2N + 1) \) infinite is similar. Let \( P \in [N] \) such that \( 2P \subset R \). We have \( \lambda \left( \bigcap_p E(s_p) \right) > 0 \). Let \( s_p = (n_{p1}^p, n_{p2}^p, \ldots, n_{p_{|p|}}^p) \) for each \( p \in P \). Let us prove that \( \{ n_k^p; p \in P \} \) is finite for all \( k \in N \). Otherwise, there exist \( k \in N, s \in N^{<N} \) and an infinite subset \( P' \) of \( P \) such that \( s_p = s^{n_k^p} t_p \) with \( t_p \in N^{<N} \) for all \( p \in P' \) and \( n_k^p \neq n_k^{p'} \) for distinct \( p, p' \in P' \). Let \( p \in P' \). For all \( x \in 2^N \), we have

\[
x \in E(s_p) \iff \begin{cases} \exists i \in [kn_k^p, k(n_k^p + 1)], x(<k, i>) = 0 \quad \text{or} \quad x \in E(s) \cap E_k(t_p) \\ \forall i \in [kn_k^p, k(n_k^p + 1)], x(<k, i>) = 1 \quad \text{and} \quad x \in F(s) \cap E_k(t_p) \end{cases}
\]

where \( E_k(t) \) and \( F_k(t) \) can be defined by induction as follows: \( E_k(\emptyset) = 2^N \) and \( F_k(\emptyset) = \emptyset \); if \( t \in N^{<N} \) has length \(|t| = j - 1 \) and \( m_j \in N \), put

\[
E_k(t \circ m_j) = \{ x \in 2^N; x \in E_k(t) \}
\quad \text{and} \quad \exists i \in [(k + j)m_j, (k + j)(m_j + 1)], x(<k + j, i>) = 0
\]
\[
or \quad (x \in F_k(t) \quad \text{and} \quad \forall i \in [(k + j)m_j, (k + j)(m_j + 1)], x(<k + j, i>) = 1 \}
\]

and

\[
F_k(t \circ m_j) = \{ x \in 2^N; x \in F_k(t) \}
\quad \text{and} \quad \exists i \in [(k + j)m_j, (k + j)(m_j + 1)], x(<k + j, i>) = 0
\]
\[
or \quad (x \in E_k(t) \quad \text{and} \quad \forall i \in [(k + j)m_j, (k + j)(m_j + 1)], x(<k + j, i>) = 1 \}
\]

Note that \( E_k(t) = F_k(t)^c \) for all \( t \in N^{<N} \). Moreover in the probability space \( (2^N, \lambda) \), the conditions \( \{ x \in E(s) \}, \{ \exists i \in [(k + j)m_j, (k + j)(m_j + 1)], x(<k + j, i>) = 0 \} \) and \( \{ x \in E_k(t) \} \) are independent, because the mappings \( x \mapsto x(j), j \in N \), are independent. The conditions \( \{ \exists i \in [kn_k^p, k(n_k^p + 1)], x(<k, i>) = 0 \} \) and \( \{ \exists i \in [kn_k^p', k(n_k^p + 1)], x(<k, i>) = 0 \} \) are also independent for distinct \( p, p' \in P' \). So we can explicitly calculate

\[
\lambda \left( \bigcap_{p \in I} E(s_p) \right) \quad \text{for any finite subset} \ I \text{ of} \ P'. \text{ We have}
\]

\[
\lambda \left( \bigcap_{p \in I} E(s_p) \right) = \sum_{i=0}^{\lfloor I \rfloor} \alpha_i (2^{-k})^i (1 - 2^{-k})^{\lfloor I \rfloor - i}
\]
where \( \alpha_0 = \lambda \left( \left[ E(s) \cap \bigcap_{p \in I} E_k(t_p) \right] \cup \left[ F(s) \cap \bigcap_{p \in I} F_k(t_p) \right] \right) \) and \( \alpha_i \geq 0 \).

\[
\sum_{i=0}^{\lvert I \rvert} \alpha_i = 1. \text{ So } \alpha_0 \leq \frac{1}{2}, \text{ whence }
\lambda \left( \bigcap_{p \in I} E(s_p) \right) \leq \frac{1}{2} (1 - 2^{-k})^{\lvert I \rvert} + \frac{1}{2} 2^{-k} (1 - 2^{-k})^{\lvert I \rvert - 1} = \frac{1}{2} (1 - 2^{-k})^{\lvert I \rvert - 1}.
\]

Thus \( \lambda \left( \bigcap_{p \in P} E(s_p) \right) = 0 \) which is a contradiction, and proves that \( \{ n_p^k; p \in P \} \) is finite for all \( k \in \mathbb{N} \). So the tree \( T' = \{ s \in \mathbb{N}^\mathbb{N}; \exists p \in P, s \text{ is an initial segment of } s_p \} \) is an infinite tree (\( P \) is infinite) with finite branching, so \( T' \not\in WF \), whence \( T \not\in WF \).

\[ \square \]

3. The abstract case.

We introduce a subset \( I \) of \( \mathcal{K}(2^\mathbb{N}) \) which plays the role of \( D \) in this simpler case.

A compact subset \( K \) of \( 2^\mathbb{N} \) is a set of type I if for all \( N \in \mathbb{N} \) there exists \( n \geq N \) such that \( x(n) = 0 \) for all \( x \in K \). Note that \( I \) is a \( G_\delta \) subset of \( \mathcal{K}(2^\mathbb{N}) \).

For each \( A \in [\mathbb{N}] \), put

\[
K_A = \{ x \in 2^\mathbb{N}; \forall n \in A, x(n) = 0 \},
\]

\[
K_A^+ = \{ x \in 2^\mathbb{N}; \exists m \in \mathbb{N}, \forall n \in A \cap [m, +\infty[, x(n) = 0 \}
\]

and let \( \mu_A \) be the Haar measure on the subgroup \( K_A \) of \( 2^\mathbb{N} \cong (\mathbb{Z}/2\mathbb{Z})^\mathbb{N} \). More precisely, \( \mu_A \) is the product measure \( \otimes_{n \in \mathbb{N}} \nu_n \) with \( \nu_n = \delta_0 \) if \( n \in A \) and \( \nu_n = \frac{1}{2} (\delta_0 + \delta_1) \) otherwise.

We will use the following elementary, but fundamental fact.

**Lemma 3.1.** — Let \( A \) and \( B \in [\mathbb{N}] \). If \( B \setminus A \) is finite, then \( \mu_A(K_B^+) = 1 \). If \( B \setminus A \) is infinite, then \( \mu_A(K_B^+) = 0 \).

Note that

\[
I = \{ K \in \mathcal{K}(2^\mathbb{N}); \exists A \in [\mathbb{N}], K \subset K_A \}
\]

and

\[
I^+ = \{ K \in \mathcal{K}(2^\mathbb{N}); \exists A \in [\mathbb{N}], K \subset K_A^+ \}.
\]
Let \( \tilde{\mu}(n) = \int x(n) \, d\mu(x) \). We have

\[
\mathcal{M}(I^\uparrow) = \{ \mu \in \mathcal{M}(2^\mathbb{N}); \ \liminf \tilde{\mu}(n) = 0 \}.
\]

Note that \( \mathcal{M}(I^\uparrow) \) is a \( \mathcal{G}_d \) subset of \( \mathcal{M}_1(2^\mathbb{N}) \).

Following Kaufman's ideas [2], we assign to each sequence \( \bar{A} = (A_n)_{n \in \mathbb{N}} \in \Sigma^\mathbb{N} \) a mapping \( A \) from \( 2^\mathbb{N} \) to \( \mathcal{P}(\mathbb{N}) \), defined by \( A(x) = \{ n \in \mathbb{N}; x \in A_n \} \), and a measure \( \nu_{\bar{A}} \) defined by \( \nu_{\bar{A}} = \int \mu_A(x) \, d\lambda(x) \). Let \( \Theta \) be the mapping from \( \Sigma^\mathbb{N} \) to \( \mathcal{M}_1(2^\mathbb{N}) \) defined by \( \Theta(A) = \nu_{\bar{A}} \). Note that \( \Theta \) is continuous.

**Lemma 3.2.** — \( \Theta(\mathcal{X}) \subset I^\perp \) and \( \Theta(\mathcal{Y}) \subset \mathcal{M}(I^\uparrow) + \mathcal{M}(I^\uparrow) \).

**Proof.** — Using Lemma 3.1 we have

\[
\lambda \left( \liminf_{R} A_n \right) = \lambda \left( \{ x \in 2^\mathbb{N}; R \setminus A(x) \text{ finite} \} \right) = \nu_{\bar{A}} \left( K_R^\uparrow \right),
\]

for all \( \bar{A} = (A_n)_{n \in \mathbb{N}} \in \Sigma^\mathbb{N} \) and \( R \in [\mathbb{N}] \). This remark allows us to finish easily the proof. \( \square \)

We have an abstract version of Theorem 1.1.

**Theorem 3.3.** — There does not exist a Borel subset \( B \) of \( \mathcal{M}_1(2^\mathbb{N}) \) such that \( \mathcal{M}(I^\uparrow) + \mathcal{M}(I^\uparrow) \subset B \) and \( B \cap I^\perp = \emptyset \).

**Proof.** — Such \( B \) insure \( (\Phi \circ \Theta)^{-1}(B) = WF^c \) and cannot be a Borel set, because \( \Phi \circ \Theta \) is continuous. \( \square \)

### 4. How to go from the abstract case to \( T \).

Every element \( x \) of \( T \) can be expressed in the form \( x = \sum_{n \in \mathbb{N}} x(n)2^{-n} \) with \( x(n) \) either 0 or 1, and \( x(n) = 0 \) for large enough \( n \) if \( x \) is rational.

For each \( A \in [\mathbb{N}] \), let

\[
K_A = \{ x \in T; \forall n \in A, x(n) = 0 \},
\]

\[
K_A^\perp = \{ x \in T; \exists m \in \mathbb{N}, \forall n \in A \cap [m, +\infty), x(n) = 0 \}
\]

and

\[
\mu_A = \bigotimes_{n \in \mathbb{N}} \left( \frac{1}{2} \delta_0 + \frac{1}{2} \delta_{2^{-n}} \right)
\]
be the canonical Bernoulli product measure concentrated on $K_A$. A set $A$ is called colacunary if for each $n \in \mathbb{N}$, there exists $a \in \mathbb{N}$ such that $[a, a + n] \subset A$. Note that $K_A \in D$ if $A$ is colacunary.

Lemma 3.1 still holds with these new notations. Our next goal is to extend this property to the $L_0$-sets.

**Lemma 4.1.** Let $K \in L_0$ and $\alpha > 0$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ witnessing this. Let $A \in [\mathbb{N}]$ and $c = \sup(-[\log_2 \alpha], 0) + 2$. If $\limsup d\left(\frac{\log_2 \varepsilon_n}{\varepsilon_n}, A\right) \geq c$, then $\mu_A(K) = 0$, where $d(x, A) = \inf \{ |x - n|; n \in A \}$ ($x \in \mathbb{R}$).

This property is derived from a result of Lyons [4] whose conclusion is much more precise, but which concerns only the case $K \in H$ and $A$ lacunary. The proof of Lemma 4.1 uses the following simple result ([1] Lemma 2.9).

**Lemma 4.2.** Let $K \in L_0$ and $\alpha > 0$ and $\varepsilon_n \in [0, \frac{1}{8}]$ witness this. Let $m = -[\log_2 \varepsilon_n]$ and $p = \sup(-[\log_2 \alpha], 0)$. For each $(x_i)_{i \in [1, m-2]} \in \{0, 1\}^{m-2}$, there exists $(x_i)_{i \in [m-1, m+p+1]} \in \{0, 1\}^{p+3}$ such that for each $x \in T$,

$$(\forall i \in [1, m + p + 1], x(i) = x_i \implies x \notin K).$$

**Proof of Lemma 4.1.** Let $K \in L_0$ and let $\alpha > 0$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ witness this. Let $p = \sup(-[\log_2 \alpha], 0)$ and $m_n = -[\log_2 \varepsilon_n]$ for each $n \in \mathbb{N}$. Without loss of generality, we can suppose that the intervals $[m_n - 1, m_n + p + 1]$, $n \in \mathbb{N}$, are pairwise disjoint and disjoint from $A$. Let $n \in \mathbb{N}$. There exists, by Lemma 4.2, a mapping $\varphi_n$ from $\{0, 1\}^{[1, m_n-2]}$ to $\{0, 1\}^{[m_n-1, m_n+p+1]}$ such that the set $B_n$ of all $x \in T$ such that

$$(\forall s \in \{0, 1\}^{[1, m_n-2]} \left(s = (x(i))_{i \in [1, m_n-2]} \implies \varphi(s) = (x(i))_{i \in [m_n-1, m_n+p+1]}\right)$$

is disjoint from $K$. But $\mu_A(B_n) = 2^{-p-3}$ and the $B_n$'s, $n \in \mathbb{N}$, are independent events in the probability space $(T, \mu_A)$, so $\mu_A(K) \leq \mu_A(\bigcap B_n^c) = \prod \mu_A(B_n^c) = 0$. \qed

### 5. Proof of Theorem 1.1.

Let $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ be two sequences of positive integers such that $\lim(b_k - a_k) = +\infty$ and $\lim(a_{k+1} - b_k) = +\infty$. Put $I_k = [a_k, b_k] \subset \mathbb{N}$.  

For $A \subset \mathbb{N}$, put $\tilde{A} = \bigcup_{k \in A} I_k$. Note that $\tilde{A}$ is colacunary if and only if $A$ is infinite.

To each sequence $\tilde{A} = (A_n)_{n \in \mathbb{N}} \in \Sigma^\mathbb{N}$, we assign a mapping $A$ from $2^\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$ defined by $A(x) = \{ n \in \mathbb{N}; x \in A_n \}$, and next a measure $\tilde{\nu}_A = \int \mu_{A(x)} d\lambda(x)$. Let $\tilde{\Theta}$ be the mapping from $\Sigma^\mathbb{N}$ to $\mathcal{M}_1(\mathbb{T})$ defined by $\tilde{\Theta}(\tilde{A}) = \tilde{\nu}_A$. Note that $\tilde{\Theta}$ is continuous.

**Lemma 5.1.** — $\tilde{\Theta}(\mathcal{X}) \subset L_0^\perp$ and $\tilde{\Theta}(\mathcal{Y}) \subset \mathcal{M}(D^\perp) + \mathcal{M}(D^\perp)$.

**Proof.** — Using Lemma 3.1, we have for each $\tilde{A} = (A_n)_{n \in \mathbb{N}} \in \Sigma^\mathbb{N}$ and each $R \in [\mathbb{N}]$,

$$
\lambda(\liminf_{R} A_n) = \lambda(\{ x \in 2^\mathbb{N}; R \setminus A(x) \text{ finite} \})
$$

$$
= \lambda(\{ x \in 2^\mathbb{N}; \tilde{R} \setminus \tilde{A}(x) \text{ finite} \})
$$

$$
= \tilde{\nu}_A(K_{R}^\perp).
$$

But $K_{R}^\perp \in D^\perp$, because $\tilde{R}$ is colacunary, whence $\tilde{\Theta}(\mathcal{Y}) \subset \mathcal{M}(D^\perp) + \mathcal{M}(D^\perp)$.

The previous remark does not allow us to prove that $\tilde{\Theta}(\mathcal{X}) \subset L_0^\perp$. Let $\tilde{A} = (A_n)_{n \in \mathbb{N}} \in \Sigma^\mathbb{N}$ such that $\tilde{\Theta}(\tilde{A}) \notin L_0^\perp$, i.e., there exists $K \in L_0$ such that $\tilde{\nu}_A(K) > 0$. Let $\alpha > 0$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ witness that $K \in L_0$. We have $\lambda(H) > 0$ with $H = \{ x \in 2^\mathbb{N}; \mu_{A(x)}(K) > 0 \}$.

Now $H \subset \{ x \in 2^\mathbb{N}; \limsup_{R} d\left( \log_2 \frac{1}{\varepsilon_n}, \tilde{A}(x) \right) < c \}$ by Lemma 4.1. Thus

$$
\limsup_{R} d\left( \log_2 \frac{1}{\varepsilon_n}, \tilde{N} \right) < c,
$$

because $\lambda(H) > 0$, so $d\left( \log_2 \frac{1}{\varepsilon_n}, \tilde{N} \right) \leq c$ for large enough $n$. Moreover $a_{k+1} - b_k > 2c$ for large enough $k$, so there exists a unique $k_n$ such that $d\left( \log_2 \frac{1}{\varepsilon_n}, I_{k_n} \right) < c$ for large enough $n (n \geq n_0)$. Let $R = \{ k_n; n \geq n_0 \}$. We have $H \subset \{ x \in 2^\mathbb{N}; R \setminus A(x) \text{ finite} \} = \liminf_{R} A_n$, so there exists $a \in \mathbb{N}$ such that $\lambda\left( \bigcap_{R \cap [a, +\infty]} A_n \right) > 0$, whence $\tilde{A} \notin \mathcal{X}$. □

Clearly, we can deduce Theorem 1.1 from this.

**6. Theorems 1.5 and 1.6 in the abstract case.**

We denote $\mathcal{C}^U = \{ X \cup Y; (X, Y) \in \mathcal{C}^2 \}$ for $\mathcal{C} \subset \mathcal{P}(E)$ where $E$ is a metrizable, compact set. It is easy to verify that

$$
\mathcal{M}(\mathcal{C}^\perp) + \mathcal{M}(\mathcal{C}^\perp) = \mathcal{M}((\mathcal{C}^\perp)^U)
$$
and

\[ M(C^1) + M(C^-) = M((C^\mu)^1). \]

We use again the notations of Part 3. Let \((A_n)_{n \in \mathbb{N}}\) be a sequence of infinite, pairwise disjoint subsets of \(\mathbb{N}\). Consider the set

\[ X_0 = \bigcup_{n \in \mathbb{N}} \bigcap_{n \geq m} (K_{A_{2n}} \cup K_{A_{2n+1}}) \]

which belongs to \((I^\mu)^1\). Note that \(X_0 \notin (I^\mu)^1\). To all \(x \in 2^\mathbb{N}\) and \(m \in \mathbb{N}\), we attach \(C_m(x) = \bigcup_{n \geq m} A_{2n+\pi(n)}\); note that \(K_{C_m(x)} \subset X_0\). Consider the weak*-integral

\[ \mu_\infty = \sum_{m \in \mathbb{N}} 2^{-m} \int \mu_{C_m(x)} \, d\lambda(x). \]

Clearly \(\mu_\infty \in M_1(X_0)\) and \(M_1(X_0) \subset M(I^\mu) + M(I^\mu)\).

**Lemma 6.1.** \(-\mu_\infty\) is not a finite sum of measures in \(M(I^\mu)\).

We can immediately deduce an abstract version of Theorem 1.5.

**Theorem 6.2.** There exists a measure in \(M(I^\mu) + M(I^\mu)\) which is not a finite sum of measures in \(M(I^\mu)\).

We can generalize the previous construction. Let \((F_m)_{m \in \mathbb{N}}\) be a sequence of finite subsets of \(\mathbb{N}\). We define

\[ \mu(F_m) = \sum_{m \in \mathbb{N}} 2^{-m} \int \mu_{C(x,F_m)} \, d\lambda(x), \]

where \(C(x,F_m) = \bigcup_{n \notin F_m} A_{2n+\pi(n)}\). Note that \(\mu(F_m) \in M_1(X_0)\). In particular, \(\mu_\infty = \mu([1,m])\).

Let \(k \in \mathbb{N}\) and \((F^k_m)_{m \in \mathbb{N}}\) be an enumeration of all subsets of \(\mathbb{N}\) containing \(k\) elements and \(\mu_k = \mu(F^k_m)\).

In particular \(\mu_1 = \mu([m])\). Note that \(\mu_k\) is concentrated on \(\bigcup_{n \in F} (K_{A_{2n}} \cup K_{A_{2n+1}})\) for each subset \(F\) of \(\mathbb{N}\) containing \(k + 1\) elements, whence \(\mu_k \in M(I^\mu)^{(2k+2)}\).

**Lemma 6.3.** \(-\mu_k \notin M(I^\mu)^{(2k+1)}\) for each \(k \geq 0\).

We can immediately deduce an abstract version of Theorem 1.6.
THEOREM 6.4. — For every $n \geq 3$, there exists a measure in $\mathcal{M}(I^\uparrow) + \mathcal{M}(I^\uparrow)$ which is the sum of $n$ measures in $\mathcal{M}(I^\uparrow)$ and is not the sum of $n-1$ measures in $\mathcal{M}(I^\uparrow)$.

PROPOSITION 6.5. — For every $n \geq 2$, there exists a measure in $\mathcal{M}(I^\uparrow)^{(n)}$ which is not in $\mathcal{M}(I^\uparrow)^{(n-1)}$.

Proof. — Consider $\nu_n = \frac{1}{n} \sum_{k=1}^{n} \mu_{A_k}$. If $B \in [\mathbb{N}]$, there exists at most one $k$ such that $B \setminus A_k$ finite, so, by Lemma 3.1, $\nu_n(K_B^\uparrow) \leq \frac{1}{n}$. If $X$ is a union of $n-1$ $I^\uparrow$-sets, then $\nu_n(X) \leq \frac{n-1}{n}$, whence $\nu_n \notin \mathcal{M}(I^\uparrow)^{(n-1)}$. □

We deduce from Theorem 6.4 and Proposition 6.5 the following fact.

COROLLARY 6.6. — The sets $\mathcal{M}(I^\uparrow)^{(n)}$ and $\mathcal{M}(I^\uparrow)^{(n)}$, $n \geq 2$, are all distinct.

We will now prove Lemmas 6.1 and 6.3.

LEMMA 6.7. — Let $\mu = \mu_{(F_m)}$ and $X \in I^\uparrow$. Then $\mu_{\uparrow X}$ is concentrated on $K_{A_p}$ for some $p$.

Proof. — Let $X \in I^\uparrow$ such that $\mu(X) > 0$. There exists $B \in [\mathbb{N}]$ such that $X \subseteq K_B^\uparrow$. If $B \setminus \bigcup_{p \in \mathbb{N}} A_p$ is infinite, then $B \setminus C(x, F_m)$ is infinite for all $m \in \mathbb{N}$ and $x \in 2^{\mathbb{N}}$. Using Lemma 3.1, we have $\mu_{C(x,F_m)}(K_B^\uparrow) = 0$, so $\mu(K_B^\uparrow) = 0$ which contradicts our hypothesis, whence $B \setminus \bigcup_{p \in \mathbb{N}} A_p$ is finite.

Consider $C = \{p \in \mathbb{N}; B \cap A_p \neq \emptyset\}$. If $C$ is infinite, $C = \{2n_k + \zeta_k; k \in \mathbb{N}, \zeta_k = 0,1\}$. If $m \in \mathbb{N}$ and $x \in 2^{\mathbb{N}}$ are such that $\mu_{C(x,F_m)}(K_B^\uparrow) > 0$, then $B \setminus C(x,F_m)$ is finite by Lemma 3.1, so $x(n_k) = \zeta_k$ for large enough $k$, because $F_m$ is finite. But $\lambda(\{x \in 2^{\mathbb{N}}; x(n_k) = \zeta_k \text{ for large enough } k\}) = 0$, so $\mu(K_B^\uparrow) = 0$. This contradiction prove that $C$ is finite and $B \cap A_p$ is infinite for some $p$.

If $m \in \mathbb{N}$ and $x \in 2^{\mathbb{N}}$ are such that $\mu_{C(x,F_m)}(K_B^\uparrow) > 0$, then $A_p \subseteq C(x,F_m)$, so $\mu_{C(x,F_m)}(K_{A_p}) = 1$, whence $\mu_{K_B^\uparrow}$ is concentrated on $K_{A_p}$. □

Proof of Lemma 6.1. — Let $X$ be a finite union of $I^\uparrow$-sets. Using Lemma 6.7, we can suppose that $X = \bigcup_{n \in F} K_{A_p}$ for some finite subset $F$. 


of $\mathbb{N}$. Let $m_0$ with $p < 2m_0$ for all $p \in F$. For all $m \geq m_0$ and $x \in 2^\mathbb{N}$, 
$\mu_{C_m(x)}(X) = 0$ by Lemma 3.1. So $\mu_\infty(X^c) > 0$. 

**Proof of Lemma 6.3.** — Using Lemma 6.7, we have just to prove that
$\mu_k$ cannot be concentrated on $X = \bigcup_{n \in F} K_{A_p}$ for every $F$ with cardinality 
$\leq 2k + 1$. Let $F$ be a set having this property. Thus $F = \{2n; n \in G\} \cup \{2n + 1; n \in G\} \cup \{2n + \zeta_n; n \in H\}$ with $\zeta_n$ either 0 or 1. Now $G$ has cardinality $\leq k$, so $G \subset F^k_m$ for some $m_0$. Using Lemma 3.1, we have 
$\mu_{C^k(x,F^k_m)}(X^c) > 0$ for every $x \in 2^\mathbb{N}$ such that $x(n) = 1 - \zeta_n$ for each 
n \in H, whence $\mu_k(X^c) > 0$. 

**7. Proof of theorems 1.5 and 1.6.**

To prove Theorems 1.5 and 1.6, we follow the ideas and techniques of Part 6. We introduce the same notations and the same lemmas, expect that, in this case, $(A_n)_{n \in \mathbb{N}}$ is a sequence of colacunary subsets of $\mathbb{N}$ such 
that for $k$ going to $+\infty$, $d(A_n \cap [k, +\infty], A_m \cap [k, +\infty]) \to +\infty$ uniformly 
for all $n \neq m$. Moreover, $K_A$ and $\mu_A, A \in [\mathbb{N}]$, are the same as in Part 4. 
Finally, Lemma 6.7 is replaced by the following result.

**LEMMA 7.1.** — Let $\mu = \mu(F_m)$ and $X \in L_0^1$. Then $\mu|_X$ is concentrated 
on $K_{A_p}$ for some $p$.

**Proof.** — We start by proving the result for $K \in L_0$. Let $\alpha > 0$ and 
$(\varepsilon_k)_{k \in \mathbb{N}}$ witness that $K \in L_0$. Let $p = \sup(-[\log_2 \varepsilon_k], 0)$, $m_k = -[\log_2 \varepsilon_k]$ 
and $J_k = [m_k - 1, m_k + p + 1], k \in \mathbb{N}$. If $\mu(K) > 0$, then $\mu_{C^k(x,F_m)}(K) > 0$ 
for some $x \in 2^\mathbb{N}$ and $m \in \mathbb{N}$. But $C(x,F_m) \subset \bigcup A_p$, so, using Lemma 4.1, 
we deduce that $J_k$ meets at least one $A_p$ for large enough $k$. Now $|J_k|$ is 
constant and as $k \to +\infty$, $d(A_n \cap [k, +\infty], A_m \cap [k, +\infty]) \to +\infty$ uniformly 
for all $n \neq m$, so $J_k$ meets exactly one $A_{p_k}$ for large enough $k$. If $(p_k)_{k \in \mathbb{N}}$ is 
unbounded, then $(p_k)_{k \in D}$ is injective for some $D \in [\mathbb{N}]$. Put $p_k = 2n_k + \zeta_k$ 
with $\zeta_k = 0$ or 1. By Lemma 4.1, if $\mu_{C(x,F_m)}(K) > 0$ for some $x \in 2^\mathbb{N}$,
then $x(n_k) = \zeta_k$ for large enough $k$. But $\lambda(\{x \in 2^\mathbb{N}; x(n_k) = \zeta_k \text{ for large enough } k\}) = 0$, so $\mu(K) = 0$. If $(p_k)_{k \in \mathbb{N}}$ is bounded, there exists $p$ such that 
$p = p_k$ for infinitely many $k$. If $m \in \mathbb{N}$ and $x \in 2^\mathbb{N}$ are such that 
$\mu_{C(x,F_m)}(K) > 0$, then $A_p \subset C(x,F_m)$, so $\mu_{C(x,F_m)}(K_{A_p}) = 1$, whence $\mu|_K$ 
is concentrated on $K_{A_p}$. 

Let $X \in L_0^+$. There exists a sequence $(K_j)_{j \in \mathbb{N}}$ of $L_0$-sets such that $X \subset \liminf K_j$. Now, for each $j$, there exists $p_j$ that $\mu_{\lfloor K_j}$ is concentrated on $K_{A_{p_j}}$, so $\mu_{\lfloor X}$ is concentrated on $\liminf K_{A_{p_j}}$. As before, $\mu(\liminf K_{A_{p_j}}) = 0$ if $(p_j)_{j \in \mathbb{N}}$ is unbounded. So $\mu_{\lfloor X}$ is concentrated on $K_{A_p}$ for some $p$. □

BIBLIOGRAPHY


Sylvain KAHANE,
Université de Paris VI
Equipe d’Analyse
4, place Jussieu
75252 Paris Cedex 05 (France).